The Theorem of Jentzsch–Szegő on an analytic curve. Application to the irreducibility of truncations of power series

Antoine Chambert-Loir

Abstract. — A theorem of Jentzsch–Szegő describes the limit measure of a sequence of discrete measures associated to zeroes of a sequence of polynomials in one variable. We extend this theorem to compact Riemann surfaces and to analytic curves over ultrametric fields. This theory is applied to the problem of irreducibility of truncations of power series with coefficients in ultrametric fields.

Résumé. — Le théorème de Jentzsch-Szegő décrit la mesure limite d'une suite de mesures discrètes associée aux zéros d'une suite convenable de polynômes en une variable. Suivant la présentation que font Andrievskii et Blatt dans [**1**], on étend ici ce résultat aux surfaces de Riemann compactes, puis aux courbes analytiques sur un corps ultramétrique. On donne pour finir quelques corollaires du cas particulier de la droite projective sur un corps ultramétrique à l'irréductibilité des polynômes-sections d'une série entière en une variable.

2000 ***Mathematics Subject Classification.*** — 11S05, 30C15, 30G06.

The construction of families of irreducible polynomials as truncations of power series with rational coefficients has attracted the attention of many mathematicians, *e.g.,* Schur [**11**], Coleman [**5**], and others. A basic example of this phenomenon is given by the exponential function,

$$\exp(T) = \sum_{j=0}^{\infty} \frac{1}{j!} T^j,$$

all of which truncations

$$f_n(T) = \sum_{j=0}^{n} \frac{1}{j!} T^j$$

are irreducible over $\mathbf{Q}$.

In his class (autumn 2009) at Princeton University, N. Katz asked whether this was a general phenomenon, *i.e.,* for general conditions on the power series which would imply irreducibility of all truncations. Alternatively, he asked for conditions which would imply reducibility. He referred to a theorem of Jentzsch [**8**] in complex analysis according to which any point of the circle of convergence is a limit point of zeroes of these truncations, provided the radius of convergence is finite and positive.

More generally, Szegő [**12**] proved that the probability measures defined by zeroes of a suitable subsequence of truncations are equidistributed on the circle of convergence. In particular, these truncations cannot be all split over $\mathbf{R}$, let alone over $\mathbf{Q}$.

Today, these theorems are understood in the context of potential theory on the Riemann sphere (see, *e.g.*, the book of Andrievskii and Blatt [**1**]).

This paper was prompted by the fact that an appropriate p-adic analogue of the Jentzsch–Szegő theorem imply stronger irreducibility properties of truncations of power series whose *p-adic* radius of convergence is finite and positive. As a corollary of our main theorem (Theorem 2.1) we obtain the following result (see Theorem 3.6). We first recall the definition of the (generalized) Tate algebras: for any positive real number R, $K\{R^{-1}T\}$ is the subalgebra of $K[[T]]$ consisting of power series $\sum a_jT^j$ such that $|a_j|\,R^j \to 0$; it is the algebra of holomorphic functions on the closed disk $E(0,R)=\{|T|\leqslant R\}$ in the sense of Berkovich.

Theorem. — *Let p be a prime number, K a finite extension of $\mathbf{Q}_p$, R a positive real number and $f=\sum_{j=0}^{\infty} a_jT^j \in K\{R^{-1}T\}$. For any nonnegative integer n, let $f_n(T)=\sum_{j=0}^{n} a_jT^j$ be the truncation of f in degree n.*

Then, for any positive integer d and any subsequence (n_k) such $|a_{n_k}|^{1/n_k} \to 1/R$, the number of K-irreducible factors of f_{n_k} of degree $\leqslant d$ is $\mathrm{o}(n_k)$. In particular, the largest degree of an irreducible factor of f_{n_k} tends to infinity for $k\to\infty$.

The classical example of the exponential series (which however does not belong to the Tate algebra $\mathbf{Q}_p\{|p|^{-1/(p-1)}\,T\}$) indicates that one cannot hope for much more in general. Indeed, the theory of Newton polygons implies that for $f=\exp(T)$, f_n has irreducible factors over $\mathbf{Q}_p$ of degrees $p, p(p-1),\dots,p^{m-1}(p-1)$, where m is the largest integer such that $p^m\leqslant n$.

Observe also that the existence of a subsequence (n_k) as in the Theorem implies that the radius of convergence of f is equal to R.

In the proofs, the restriction to elements of a Tate algebra is essential. We explain in Remark 3.2 why this is a major defect for the application to irreducibility. However, an easy construction (Example 3.7) shows that the theorem does not extend to arbitrary power series.

This article has three parts. First we generalize the methods of Andrieveskii and Blatt to include compact Riemann surfaces of arbitrary genus, see Theorem 1.2. The main interest of this extension is to prepare the second part where we prove an analogue of the Jentzsch–Szegő theorem in the ultrametric setting, *i.e.,* when the compact Riemann surface is replaced by a smooth projective analytic curve in the sense of Berkovich [**3**]. The non-archimedean potential theory developed by Thuillier [**13**] and Baker/Rumely [**2**] is formally identical to the classical complex potential theory. In particular, the proof of Section 1 applies verbatim. In Section 3, we apply this to the Berkovich projective line and deduce our main results concerning irreducible factors of truncations of power series over a locally compact ultrametric field.

Acknowledgments. — This paper was written during a stay at the Institute for Advanced Study in Princeton. I am very grateful to Nick Katz for asking this question, which revived my interest in potential theory on ultrametric curves, and to Enrico

Bombieri for many discussions on related questions. The idea of extending the theorem of Jentzsch–Szegő to curves of arbitrary genus is due to Antoine Ducros; I would like to thank him, Amaury Thuillier, Matt Baker, and Robert Rumely for many stimulating exchanges concerning potential theory. I would also like to thank the referee for his/her careful reading.

I was partially supported by Institut universitaire de France, and the National Science Foundation grant DMS-0635607.

1. Riemann surfaces

Let M be a compact connected Riemann surface. Let p be a point of M and E a compact non-polar subset of M; we assume that $\Omega = M \setminus E$ is connected and contains p. Fix a local parameter z in a neighborhood of p. The Green function G_E is the unique subharmonic function on $M \setminus \{p\}$ such that

(1) it vanishes on E (up to a polar subset of ∂E),

(2) it is harmonic on $M \setminus (\{p\} \cup E)$

(3) and it has an expansion around p of the form

$$G_E(q) = \log \left| z(q)^{-1}/\operatorname{cap}(E) \right| + \mathrm{o}(1).$$

The positive real number $\operatorname{cap}(E)$ is the capacity of E with respect to p, relative to the local parameter z. More intrinsically, there exists a norm $\|\cdot\|^{\mathrm{cap}}$ on the complex tangent line T_pM such that $\operatorname{cap}(E)$ is the norm of the tangent vector $\partial/\partial z \in T_pM$. This norm does not depend on the choice of the local parameter z. The equilibrium measure of E is the probability measure

$$\mu_E = \mathrm{dd^c}\, G_E + \delta_p;$$

it is supported on the boundary $\partial E = E \setminus \mathring{E}$ of E. Finally, for any function f on M, we define $\|f\|_E = \sup_E |f|$. If f is holomorphic on a neighborhood of E, then $\|f\|_E = \sup_{\partial E} |f|$ (maximum principle).

Let k be a positive integer and $f \in \Gamma(M, \mathscr{O}(kp))$, a meromorphic function on M, holomorphic on $M \setminus \{p\}$ with a pole of order $\leqslant k$ at p. Its leading coefficient at p, $j^k(f)$, is defined as

$$j^k(f) = \lim_{q \to p} f(q) z(q)^k \left(\frac{\partial}{\partial z}\right)^{\otimes k};$$

it is an element of $T_pM^{\otimes k}$, independent of the choice of the local parameter z, and vanishing if and only if the order of the pole of f at p is $< k$.

Lemma 1.1. — *Let k be a positive integer and let f be any non-zero element of $\Gamma(M, \mathscr{O}(kp))$. The function $\frac{1}{k} \log |f| - G_E$ is subharmonic on Ω. For any point $q \in M \setminus \{p\}$, one has*

$$|f(q)| \leqslant \|f\|_E \exp(kG_E(q)).$$

In particular,

$$\left\|j^k(f)\right\|^{\mathrm{cap}} \leqslant \|f\|_E .$$

Proof. — Set $\varphi = \frac{1}{k}\log|f| - G_E$. The function φ is subharmonic on $\Omega \setminus \{p\}$, since on this set, $\log|f|$ is subharmonic and G_E is harmonic. In fact, it is subharmonic on Ω since, after choosing a local parameter z at p, $q \mapsto f(q)z(q)^k$ is holomorphic in a neighborhood of p. By the maximum principle for subharmonic functions of [**14**] (Theorem III.28, p. 77), we have

$$\sup_\Omega \varphi = \sup_{q\in\partial\Omega} \operatorname{fine.lim}_{\substack{z\to q\\ z\in\Omega}} \varphi(z) = \sup_{q\in\partial\Omega} \frac{1}{k}\log|f(q)| = \frac{1}{k}\log\sup_{\partial E}|f| = \frac{1}{k}\log\|f\|_E .$$

(Taking limits for the fine topology, we may ignore the eventual polar subset of ∂E at which G_E does not tend to 0.) Moreover,

$$\begin{aligned}\varphi(p) &= \frac{1}{k}\lim_{q\to p}\varphi(q) = \frac{1}{k}\lim_{q\to p}\log\left|f(q)z(q)^k\right| - \lim_{q\to p}\Big(G_E(q) - \log|z(q)|^{-1}\Big)\\ &= \frac{1}{k}\log\left\|j^k(f)\right\|^{\mathrm{cap}} .\end{aligned}$$

Consequently, $\left\|j^k(f)\right\|^{\mathrm{cap}} \leqslant \|f\|_E$. ☐

For such a function f, let $\nu(f)$ be the measure $f^*\delta_0/k$ given by the zeroes of f (divided by k). It is a positive measure on M with total mass $\leqslant 1$, and a probability measure if and only if $j^k(f) \neq 0$.

Theorem 1.2. — *Let (k_n) be a sequence of positive integers. For any n, let $f_n \in \mathscr{O}(k_n p)$ be a non-zero meromorphic function on M with a pole of order at most k_n at p. Assume that:*

(1) $\overline{\lim}_n \dfrac{1}{k_n}\log\|f_n\|_E \leqslant 0$;

(2) *for any compact subset C in $\mathring{E}$, $\nu(f_n)(C) \to 0$;*

(3) *there exists a non-empty compact subset S in Ω such that*

$$\underline{\lim}_n \sup_S \Big(\frac{1}{k_n}\log|f_n| - G_E\Big) \geqslant 0.$$

Then, the sequence of measures $(\nu(f_n))$ converges to the equilibrium measure μ_E in the weak-$*$ topology.

Remarks 1.3. a) For any n, the function $\varphi_n = k_n^{-1}\log|f_n| - G_E$ is subharmonic on Ω. In particular, it is upper-semicontinuous, hence bounded from above on any compact subset of Ω. The upper-bound on S in condition (3) is therefore finite. More precisely, we have seen that $\sup_S \varphi_n \leqslant k_n^{-1}\log\|f_n\|_E$. Condition (3) implies that $\underline{\lim}_n k_n^{-1}\log\|f_n\|_E \geqslant 0$. Condition (1) implies $\overline{\lim}_n \sup_S \varphi_n \leqslant 0$. The conjunction of Conditions (1) and (3) is thus equivalent to the two equalities

$$\lim_n \frac{1}{k_n}\log\|f_n\|_E = \lim_n \sup_S \varphi_n = 0.$$

b) For $S = \{p\}$, Condition (3) is equivalent to

$$\varliminf_n \frac{1}{k_n} \log \|j_{k_n}(f_n)\|^{\mathrm{cap}} \geqslant 0$$

but requiring this inequality is more restrictive. For example, if $f_n \in \mathscr{O}((k_n-1)p)$, then $j^k(f_n) = 0$ but Condition (3) still can be valid for some compact subset. When Condition (3) holds for $S = \{p\}$, Condition (1) implies that

$$\lim \frac{1}{k_n} \log \|j_{k_n}(f_n)\|^{\mathrm{cap}} = 0.$$

Lemma 1.4. — *Let (k_n) be a sequence of positive integers. For any n, let $f_n \in \mathscr{O}(k_n p)$. Assume that Conditions (1) and (3) of Theorem 1.2 hold. Then, for any compact and non-polar subset $T \subset \Omega$, one has*

$$\varliminf_n \sup_T \Big(\frac{1}{k_n} \log |f_n| - G_E\Big) = 0.$$

Proof. — Set $\varphi_n = \frac{1}{k_n} \log |f_n| - G_E$ and let S be a non-empty compact subset of Ω such that $\varliminf_n \sup_S \varphi_n \geqslant 0$. Let $m = \varliminf_n \sup_T \varphi_n$. By Remark 1.3, a), it suffices to prove that $m \geqslant 0$.

First assume that T is disjoint from S. Then there exists a harmonic function u on $\Omega \setminus T$ which, up to a set of capacity zero, vanishes on the boundary of E and equals m at the boundary of T. Let $\varepsilon > 0$; by Remark 1.3, a), for sufficiently large integers n, $\varphi_n \leqslant u + \varepsilon$ on ∂E (Condition (1)), as well as on ∂T (by the definition of m), modulo subsets of zero capacity. Since φ_n is subharmonic on $\Omega \setminus T$ the maximum principle of [**14**] (Theorem III.28, p. 77) implies that $\varphi_n \leqslant u+\varepsilon$ on $\Omega \setminus T$. Therefore, $\sup_S \varphi_n \leqslant \sup_S u+\varepsilon$ and $\varliminf_n \sup_S \varphi_n \leqslant \sup_S u+\varepsilon$. Considering arbitrary small positive ε, we obtain $\varliminf_n \sup_S \varphi_n \leqslant \sup_S u$. If $m < 0$, the strong maximum principle implies that $u < 0$ on $\Omega \setminus T$ (since Ω is connected, the closure of any connected component of $\Omega \setminus T$ meets T), hence $\sup_S u < 0$, a contradiction.

In general, let T' be a compact non-polar subset of Ω, disjoint from $S \cup T$; for example, a closed disk (of non-empty interior) contained in the complementary subset. By the previous case, the statement holds for T' (since T' is disjoint from S). Since T is disjoint from T', it also holds for T. ☐

Lemma 1.5. — *Under the assumptions of Theorem 1.2, the sequence (k_n) converges to $+\infty$.*

Proof. — Assume otherwise. Choosing a subsequence if necessary, we may assume that the sequence (k_n) is constant, equal to a positive integer k. Then, $\lim \log \|f_n\|_E = 0$; in particular, the sequence $(\|f_n\|_E)$ is bounded. Since E is infinite (being non-polar), $\|\cdot\|_E$ is a norm on $\Gamma(M, \mathscr{O}(kp))$. Since this space is finite-dimensional, all norms on it are equivalent, and the sequence (f_n) contains a converging subsequence. Its limit is a function $f \in \mathscr{O}(kp)$. The convergence is uniform on any compact subset of $M \setminus \{p\}$. By Condition (2) and Hurwitz's Theorem, f does not vanish on $\mathring{E}$.

Let S be a compact and non-polar subset of $\Omega \setminus \{p\}$; By Lemma 1.4, we have

$$0 \leqslant \varliminf_n \sup_S \left(\frac{1}{k_n} \log |f_n| - G_E \right) = \sup_S \left(\frac{1}{k} \log |f| - G_E \right).$$

In particular, $\sup_S |f| \geqslant e^{kG_E} \geqslant 1$. Since S is arbitrary, we conclude that $|f(q)| \geqslant e^{kG_E(q)} \geqslant 1$ for any $q \in \Omega \setminus \{p\}$. This implies that f doesn't vanish on $\Omega \setminus \{p\}$ and that the order of its pole at p is equal to k. Letting the point q tend to a point of ∂E, we see that $|f| \geqslant 1$ on ∂E.

In conclusion, f doesn't vanish on $M \setminus \{p\}$, which contradicts the presence of a pole at p. □

Lemma 1.6. — *Under the assumptions of Theorem 1.2, any limit measure ν of the sequence $(\nu(f_n))$ is a probability measure supported on ∂E.*

Proof. — First of all, ν is a probability measure. Indeed, let d_n be the order of the pole of f_n at p; it is the smallest integer d such that $f \in \mathscr{O}(dp)$ and the mass of $\nu(f_n)$ equals d_n/k_n. Set

$$\varphi'_n = \frac{1}{k_n} \log |f_n| - \frac{d_n}{k_n} G_E.$$

This is a subharmonic function on Ω, bounded from above by $\frac{1}{k_n} \log \|f_n\|_E$. Moreover, since

$$\varphi_n = \frac{1}{k_n} \log |f_n| - G_E = \varphi'_n - \frac{k_n - d_n}{k_n} G_E,$$

one has

$$\sup_S \varphi_n \leqslant \sup_S \varphi'_n - \frac{k_n - d_n}{k_n} \inf_S G_E \leqslant \frac{1}{k_n} \log \|f_n\|_E - \frac{k_n - d_n}{k_n} \inf_S G_E,$$

where S is any compact and non-polar subset of Ω, disjoint from p, so that $\inf_S G_E > 0$. When $n \to \infty$, it follows that

$$0 \leqslant \varliminf_n \sup_S \varphi_n \leqslant - \varlimsup_n \frac{k_n - d_n}{k_n} \inf_S G_E.$$

Consequently, $\varlimsup_n \frac{k_n - d_n}{k_n} \leqslant 0$, hence

$$\lim_n \frac{d_n}{k_n} = 1.$$

We now show that the support of ν is contained in ∂E. By Condition (2), it is disjoint from $\mathring{E}$. Thus it suffices to prove that it is contained in E.

Let C be a compact subset in Ω. We claim that $\nu(f_n)(C) \to 0$. Indeed, for $c \in C$, let $G_{E,c}$ be the Green function for E with pole at c. For any integer n, let $(c_{n,j})_{j \in J_n}$ be the family of those zeroes of f_n which belong to C, repeated according to their multiplicity. Set

$$\varphi'_n = \varphi_n + \frac{1}{k_n} \sum_{j \in J_n} G_{E, c_{n,j}};$$

it is a subharmonic function on Ω, bounded from above by $\frac{1}{k_n}\log\|f_n\|_E$. Let S be a compact and non-polar subset of Ω, disjoint from C. Since the function $(q,c)\mapsto G_{E,c}(q)$ is continuous and positive on $S\times C$, it follows that $\inf_S \inf_{c\in C} G_{E,c} > 0$. Then,

$$\sup_S \varphi'_n \geqslant \sup_S \varphi_n + \inf_S \frac{1}{k_n}\sum_{j\in J_n} G_{E,c_{n,j}} \geqslant \sup_S \varphi_n + \nu(f_n)(C) \inf_S \inf_{c\in C} G_{E,c},$$

and

$$\varliminf_n \sup_S \varphi'_n \geqslant \varliminf_n \nu(f_n)(C) \inf_{c\in C} G_{E,c}.$$

On the other hand, the inequality $\varphi'_n \leqslant \frac{1}{k_n}\log\|f_n\|_E$ implies that

$$\varliminf_n \sup_S \varphi'_n \leqslant 0.$$

It follows that $\varliminf_n \nu(f_n)(C) = 0$, as claimed. Passing to subsequences, this implies that $\nu(\mathring{C}) = 0$. Since Ω is locally compact, we conclude that the support of ν is disjoint from Ω, so is contained in E. This concludes the proof. □

Proof of Theorem 1.2. — Since M is compact, the space of probability measures on M is also compact. It suffices to prove that μ_E is the only possible limit value of the sequence $(\nu(f_n))$. Let ν be such a limit value. By Lemma 1.6, ν is a probability measure supported on ∂E. Replacing k_n by the order of the pole of f_n at p, we suppose that for any n, $f_n \notin \mathscr{O}((k_n-1)p)$, in other words, $j^k(f_n)\neq 0$.

Since ν admits a countable basis of neighborhoods, and after passing to a subsequence, we may assume that $\nu(f_n)$ converges to ν.

Let $g(\cdot,\cdot)_p =$ be a Green kernel on $M\times M$ relative to the point p; this is a distribution on $M\times M$ satisfying the following properties:

- the partial differential relation $\mathrm{dd}^c g + \delta_\Delta = \delta_{p\times M} + \delta_{M\times p}$ holds;
- the distribution g is symmetric.

If $M = \mathbf{P}^1$, $p=\infty$ and $M\setminus\{p\}$ is identified with $\mathbf{C}$, one can take $g(z_1,z_2)_p = \log|z_1-z_2|^{-1}$. We refer to [**10**] for a proof of the existence of such a distribution in general. (In the notation of that book, this is the distribution $-\log[\cdot,\cdot]_p$.) Moreover, for any points m and m' in $M\setminus\{p\}$, one has

$$\lim_{z\to p}\,(g(z,m)_p - g(z,m')_p) = 0, \tag{1.7}$$

uniformly when m and m' belong to a fixed compact subset of $M\setminus\{p\}$. The Green kernel thus defines a local parameter at p, well-defined up to multiplication by a local holomorphic function of absolute value equal to 1 at p.

For any measure α on M whose support does not contain p, let

$$U^\alpha = \int g(z,\cdot)_p \mathrm{d}\alpha(z)$$

be the potential of α with respect to the kernel g; it is a distribution on M such that $\mathrm{dd}^c U^\alpha + \alpha = \|\alpha\|\,\delta_p$, where $\|\alpha\| = \langle\alpha,1\rangle$ is the total mass of α. In particular, U^α is subharmonic outside of the support of α. If α is positive, U^α is subharmonic outside of p. If the total mass of α is zero then Equation (1.7) implies that U^α is continuous

and vanishes at p. The Green function G_E with pole at p can be written in terms of the potential U^{μ_E} of the equilibrium measure μ_E: one has $U^{\mu_E} = -G_E - \log \operatorname{cap}(E)$.

For any meromorphic function $f \in \mathscr{O}(kp)$ such that $j^k(f) \neq 0$, $\log|f| + kU^{\nu(f)}$ is a harmonic function on the compact space M, hence constant. Let a be a complex number such that this function equals $\log|a|$. Then,

$$\varphi = \frac{1}{k}\log|f| - G_E = -U^{\nu(f)} - G_E + \frac{1}{k}\log|a| = -U^{\nu(f)} + U^{\mu_E} + \frac{1}{k}\log\left\|j^k(f)\right\|^{\mathrm{cap}}$$

since $U^{\nu(f)} - U^{\mu_E}$ vanishes at p.

To prove Theorem 1.2, we establish the inequality $U^\nu \leqslant U^{\mu_E}$. Then a standard argument shows that $\mu_E = \nu$.

Let V a compact non-polar subset of Ω and W a compact neighborhood of V contained in Ω. Assume that $p \notin V$ but $p \in W$. The measure $\nu(f_n)$ splits canonically as the sum $\nu_W(f_n) + \nu^W(f_n)$ of two measures, where $\nu_W(f_n) = \nu(f_n)\mathbf{1}_W$ is supported on W, while W has measure 0 with respect to $\nu^W(f_n) = \nu(f_n)(\mathbf{1} - \mathbf{1}_W)$.

According to Lemma 1.6, $\|\nu_W(f_n)\|$ tends to 0 when n goes to $+\infty$, so that ν is also a limit value of the sequence $(\nu^W(f_n))$. Since $g(\cdot,\cdot)_p$ is bounded on $\complement W \times V$, U^ν is a limit value of the sequence $(U^{\nu^W(f_n)})$, for the topology of uniform convergence on V. (For any subset A of M, we write $\complement M$ to denote the complementary subset $M \setminus A$ to A in M.)

Following Andrievskii and Blatt [**1**], let us decompose

$$\begin{aligned}
\varphi_n &= \frac{1}{k_n}\log|f_n| - G_E \\
&= -U^{\nu(f_n)} + U^{\mu_E} + \frac{1}{k_n}\log\left\|j^k(f_n)\right\|^{\mathrm{cap}} \\
&= \left(-U^{\nu^W(f_n)} + \left\|\nu^W(f_n)\right\| U^{\mu_E}\right) + \|\nu_W(f_n)\|\, U^{\mu_E} \\
&\qquad + \left(\frac{1}{k_n}\log\left\|j^k(f_n)\right\|^{\mathrm{cap}} - U^{\nu_W(f_n)}\right).
\end{aligned}$$

The function $-U^{\nu_W(f_n)}$ is subharmonic on $M \setminus \{p\}$. Let R be any compact neighborhood of p, disjoint from V and contained in W. Since $V \subset \complement R \subset M \setminus \{p\}$, the maximum principle implies that

$$\sup_V(-U^{\nu_W(f_n)}) \leqslant \sup_{\complement R}(-U^{\nu_W(f_n)}) = \sup_{\partial R}(-U^{\nu_W(f_n)}).$$

Consequently,

$$\begin{aligned}
&\sup_V\left(\frac{1}{k_n}\log\left\|j^k(f_n)\right\|^{\mathrm{cap}} - U^{\nu_W(f_n)}\right) \\
&\qquad \leqslant \sup_{\partial R}\varphi_n - \inf_{\partial R}\left(-U^{\nu^W(f_n)} + \left\|\nu^W(f_n)\right\| U^{\mu_E}\right) - \|\nu_W(f_n)\| \inf_{\partial R} U^{\mu_E}.
\end{aligned}$$

The function $-U^{\nu^W(f_n)} + \left\|\nu^W(f_n)\right\| U^{\mu_E}$ is continuous on R, and harmonic on $\mathring{R}$. For n going to infinity, it converges uniformly to $-U^\nu + U^{\mu_E}$, which is again continuous on R, harmonic on $\mathring{R}$ and vanishing at p. It follows that

$$\lim_n \inf_{\partial R} \left(-U^{\nu^W(f_n)} + \left\|\nu^W(f_n)\right\| U^{\mu_E}\right) = \inf_{\partial R} \left(-U^\nu + U^{\mu_E}\right).$$

By Lemma 1.1 and Condition (1) of Theorem 1.2, $\varphi_n \leqslant \frac{1}{k_n} \log \|f_n\|_E$ and

$$\overline{\lim_n} \sup_{\partial R} \varphi_n \leqslant 0.$$

Since $\|\nu_W(f_n)\|$ tends to 0,

$$\lim_n \|\nu_W(f_n)\| \inf_{\partial R} U^{\mu_E} = 0.$$

Finally,

$$\overline{\lim_n} \sup_V \left(\frac{1}{k_n} \log \left\|j^k(f_n)\right\|^{\mathrm{cap}} - U^{\nu_W(f_n)}\right) \leqslant \sup_{\partial R} \left(U^\nu - U^{\mu_E}\right).$$

Choose the compact neighborhood R arbitrarily close to $\{p\}$. Since $U^\nu - U^{\mu_E}$ is continuous and vanishes at p we deduce

$$\overline{\lim_n} \sup_V \left(\frac{1}{k_n} \log \left\|j^k(f_n)\right\|^{\mathrm{cap}} - U^{\nu_W(f_n)}\right) \leqslant 0. \tag{1.8}$$

Furthermore,

$$\begin{aligned}
\inf_V U^{\nu^W(f_n)} &= \inf_V \left(U^{\nu(f_n)} - U^{\nu_W(f_n)}\right) \\
&= \inf_V \left(\left(U^{\nu(f_n)} - U^{\mu_E}\right) - \left(U^{\nu_W(f_n)} - U^{\mu_E}\right)\right) \\
&\leqslant \inf_V \left(U^{\nu(f_n)} - U^{\mu_E}\right) - \inf_V U^{\nu_W(f_n)} + \sup_V U^{\mu_E}.
\end{aligned}$$

Since $U^{\nu^W(f_n)}$ converges uniformly to U^ν on V, Equation (1.8) implies that

$$\inf_V U^\nu \leqslant \overline{\lim_n} \inf_V \left(U^{\nu(f_n)} - U^{\mu_E} - \frac{1}{k_n} \log \left\|j^{k_n}(f_n)\right\|^{\mathrm{cap}}\right) + \sup_V U^{\mu_E}.$$

However

$$U^{\nu(f_n)} - U^{\mu_E} - \frac{1}{k_n} \log \left\|j^{k_n}(f_n)\right\|^{\mathrm{cap}} = -\frac{1}{k_n} \log |f_n| + G_E,$$

so that

$$\overline{\lim_n} \inf_V \left(U^{\nu(f_n)} - U^{\mu_E} - \frac{1}{k_n} \log \left\|j^{k_n}(f_n)\right\|^{\mathrm{cap}}\right) \leqslant -\underline{\lim_n} \sup_V \left(\frac{1}{k_n} \log |f_n| - G_E\right) \leqslant 0$$

by Lemma 1.4.

This proves the inequality

$$\inf_V U^\nu \leqslant \sup_V U^{\mu_E}.$$

Since U^ν and U^{μ_E} are continuous on $\complement(E \cup \{p\})$, this implies that $U^\nu \leqslant U^{\mu_E}$ on $\complement(E \cup \{p\})$.

It follows that $\nu = \mu_E$. Indeed, since U^ν is subharmonic outside E and U^{μ_E} is bounded from above by $-\log \operatorname{cap}(E) = I(\mu_E)$ on ∂E, up to a polar subset, we have

$U^\nu \leqslant I(\mu_E)$ on ∂E. Then, the energy $I(\nu) = \int U^\nu \,\mathrm{d}\nu$ of ν is bounded from above by $I(\mu_E)$. Since μ_E is the unique measure of minimal energy supported on E, we obtain that $\nu = \mu_E$, as claimed. □

2. Analytic curves over ultrametric fields

Let K be a complete ultrametric valued field (of any characteristic). Let M a smooth projective, geometrically connected curve over K; let $p \in M(K)$ and let z be a local parameter at p.

We view M as a K-analytic curve in the sense of Berkovich [**3**]. Recall that $M \setminus \{p\}$ is the Berkovich spectrum $\mathscr{M}(A)$ of the K-algebra $A = \Gamma(M \setminus \{p\}, \mathscr{O}_M)$, *ie.*, the set of multiplicative seminorms on this K-algebra which extend the absolute value of K, endowed with the coarsest topology for which all maps $a \mapsto (x \mapsto |a|(x))$ are continuous. We use the standard notation in this subject: if $x \in \mathscr{M}(A)$ and $a \in A$, $|a|(x)$ is the value at a of the semi-norm x. Every K-rational point of M defines a canonical element of M; if $q \neq p$, this is just the semi-norm $a \mapsto |a(q)|$ on A. By [**3**], the space M is connected, locally contractible and compact. If K admits a countable dense subset, the space M is also metrizable.

By the works of Favre/Jonsson [**6**], Favre/Rivera-Letelier [**7**], Thuillier [**13**], Baker/Rumely [**2**], it is well-known that such a space admits a potential theory formally analogous to that on compact Riemann surfaces. Therefore, all statements of the first Section, and their proofs, translate directly to the ultrametric setting.

When M is the projective line, the required theory is the subject of the book [**2**] by Baker and Rumely. In his unpublished PhD Thesis, Thuillier [**13**] developed a more general theory, valid for arbitrary curves. Here, we recall briefly the main aspects of his theory.

The Berkovich space M carries two sheaves, the sheaf $\mathscr{A}$ of smooth functions, and its subsheaf $\mathscr{H}$ of harmonic functions; both are subsheaves of the sheaf of real valued continuous functions on M. There is a notion of subharmonic functions; these obey a maximum principle. If U is an open subset of M and $f \in \mathscr{O}(U)$ is an analytic function on U, the function $\log|f|$ is subharmonic, and is harmonic if f doesn't vanish. Harmonic functions satisfy Harnack's principle; in particular, a uniform limit of harmonic functions on an open set is harmonic.

Furthermore, one defines sheaves of smooth forms, distributions $\mathscr{D}^0$, and currents $\mathscr{D}^1$, as well as a Laplace operator $\mathrm{dd}^c\colon \mathscr{D}^0 \to \mathscr{D}^1$. Smooth forms are locally finite linear combinations of Dirac measures at points of type II or III, distributions are dual to smooth forms, currents are dual to smooth functions; there are canonical inclusions of smooth functions into distributions, and of smooth forms into currents.

If u is smooth, $\mathrm{dd}^c u$ is a smooth form. A function u on an open subset U of M is subharmonic if and only if $\mathrm{dd}^c u$ is a positive measure on U. Furthermore, a Radon measure μ on M is of the form $\mathrm{dd}^c T$, for some distribution T, if and only if $\langle \mu, 1\rangle = 0$.

The Dirichlet problem on an open subset of M is solved using Perron's method. Barriers exist at any point of M which is not of type I. This implies existence

and uniqueness of a Green function G_E for a compact subset E, with a pole at a prescribed K-point $p \notin E$.

In classical potential theory, or Abstract potential theory (see, *eg.*, [**9**] and [**4**]) the *kernel* plays an important rôle. It is an upper-continuous function on the product space $(M \setminus p) \times (M \setminus p)$. For $M = \mathrm{P}^1$ and $p = \infty$, this is the so-called *Hsia kernel* $\delta(\cdot,\cdot)_\infty$, or rather $-\log \delta(\cdot,\cdot)_\infty$. Baker and Rumely describe it in detail in [**2**, Chapter 4]. In general, Thuillier sketches a construction of this kernel in Chapter 5 of [**13**]: for $m, m' \in M \setminus p$, $g(m,m') = g_m(m')$, where g_m is the unique continuous function on M, with values in $\mathbf{R} \cup \{\pm\infty\}$, solution of the equation $\mathrm{dd}^c\, g_m = \delta_m - \delta_p$ which admits an expansion

$$g_m(q) = \log |z(q)| + \mathrm{o}(1)$$

in a neighborhood of p. This function g is symmetric, continuous with respect to each variable, lower semi-continuous, and even continuous outside the diagonal. Moreover, this kernel g is the largest semi-continuous extension of the kernel $-\log[\cdot,\cdot]_p$ constructed by Rumely in his book [**10**].

If μ is a measure with compact support in $M \setminus p$, its potential U^μ is the unique solution of the distribution equation

$$\mathrm{dd}^c\, U^\mu = \mu - \langle \mu, 1\rangle \delta_p$$

satisfying

$$U^\mu(q) = \langle \mu, 1\rangle \log |z(q)| + \mathrm{o}(1)$$

in a neighborhood of p. It can be computed using the kernel, by the formula

$$U^\mu(m) = \int_{M\setminus p} g(m,m')\, \mathrm{d}\mu(m').$$

For $M = \mathrm{P}^1$, the maximum and continuity principles, analogs to theorems of Maria and Frostman, are proved by Baker and Rumely ([**2**], Theorems 6.15 and 6.18). In general, one can refer to Abstract potential theory. By [**4**], the maximum and continuity principles are satisfied as soon as subharmonic functions satisfy the maximum principle, which is the case. (Note that Brelot's axiomatic in [**4**] only considers positive kernels. However, since we will only look at measures whose support is compact in $M \setminus p$, the required assertions remain true, with essentially the same proofs.)

The energy of a measure μ with compact support in $M \setminus p$ is given by the formula

$$I(\mu) = \int_{M\setminus p} U^\mu(m)\, \mathrm{d}\mu(m) = \int_{(M\setminus p)^2} g(m,m')\, \mathrm{d}\mu(m)\, \mathrm{d}\mu(m').$$

Robin's constant $V_p(E)$ of a compact subset E of $M \setminus p$ is the lower bound of the energies of probability measures supported on E. If E is not polar, that is, if $V_p(E) \neq +\infty$, there exists a unique probability measure μ_E supported on E such that $I(\mu_E) = V_p(E)$: this is the *equilibrium measure* of E. The existence of equilibrium measures is a consequence of compactness of the space of probability measures on E. For $M = \mathrm{P}^1$, uniqueness is shown in Prop. 7.21 of [**2**], relying on a *strong maximum principle* (Prop. 7.17). Theorem 3.6.11 in Thuillier's [**13**] furnishes

the "Evans functions" used by Baker and Rumely in their proof, so that existence and uniqueness of an equilibrium measure holds in general.

In Section 1, we had to extract converging subsequences of sequences of probability measures. This is still possible when the field K admits a countable dense subset since, in that case, the space M and the space of probability measures on M are compact and metrizable. In the general case, subsequences may not suffice but it suffices to carry out the arguments using ultrafilters instead of subsequences. Alternatively, one can also replace sequences by nets, as Baker and Rumely do in [**2**].

It is now clear that the arguments given in Section 1 to prove Theorem 1.2 translate in the present setting of analytic curves over ultrametric fields and furnish the following theorem.

Theorem 2.1. — *Let K be a complete valued ultrametric field, M a projective smooth and geometrically connected curve over K, viewed as a K-analytic curve in the sense of Berkovich. Let p be a K-rational point of M, z a local parameter at p. Let E be a compact non-polar subset of $M \setminus \{p\}$ such that $\Omega = M \setminus E$ is connected.*

For any non-zero rational function f on M, let $\nu(f)$ be the probability measure on M given by

$$\nu(f) = \frac{1}{\deg(f)} \sum_{f(q)=0} \operatorname{ord}_q(f)\delta_q,$$

where the sum is over the zeroes of f; we also set $\|f\|_E = \sup_{\partial E} |f|$.

Let (k_n) be a sequence of positive integers. For any n, let $f_n \in \mathscr{O}(k_n p)$ be a non-zero *meromorphic function on M having a pole of order at most k_n at p. Let us make the following assumptions:*

(1) $\varlimsup_n \frac{1}{k_n} \log \|f_n\|_E \leqslant 0$*;*

(2) *for any compact subset C in $\mathring{E}$, $\nu(f_n)(C) \to 0$;*

(3) *there exists a non-empty compact subset S in Ω such that*

$$\varliminf_n \sup_S (\frac{1}{k_n} \log |f_n| - G_E) \geqslant 0.$$

Then, the sequence of measures $(\nu(f_n))$ converges to the equilibrium measure μ_E for the weak-$*$ topology.

3. Applications to irreducibility

Let K be a complete ultrametric valued field (of any characteristic). Let $M = \mathrm{P}^1$ be the projective line over K in the sense of Berkovich. Let $p \in M$ be its point at infinity. The space $M \setminus p$ is the analytic affine line A^1, that is, the Berkovich spectrum $\mathrm{M}(K[T])$ of the polynomial algebra $K[T]$. Let us fix T^{-1} as a local parameter at p.

Let R be a positive real number. The closed disk, denoted $E(0,R)$ by Berkovich, is the set of points x in A^1 such that $|T|(x) \leqslant R$. It is a compact subset in A^1 whose Shilov boundary has a unique point $\xi(R)$; in other words, any holomorphic function on $E(0,R)$ reaches its maximum at $\xi(R)$; this point is the Gauß seminorm

$$P = \sum_{j=0}^{\infty} a_j T^j \mapsto \max_j |a_j| R^j;$$

the multiplicativity of this seminorm is Gauß's theorem. We also write $\|\cdot\|_R$ for the supremum norm of a polynomial or an analytic function on the disk $E(0,R)$. The interior of $E(0,R)$ in the affine line A^1 is equal to $E(0,R) \setminus \{\xi(R)\}$ ([**3**], Corollary 2.5.13). The open disk $D(0,R)$ is the set of points x such that $|T|(x) < R$.

The Green function for $E(0,R)$ (with pole at infinity) is given by $x \mapsto \log\max(|T|(x)/R, 1)$; its equilibrium measure is the Dirac measure at $\xi(R)$. As a particular case of Theorem 2.1, we obtain:

Proposition 3.1. — *Let us consider a sequence of polynomials (f_n) satisfying the following properties:*

(1) *the degree k_n of f_n tends to $+\infty$;*

(2) *the sequence (f_n) converges uniformly on the disk $E(0,R)$ to a non zero function.*

(3) *the sequence (a_n) given by the leading coefficient a_n of f_n satisfies $\lim |a_n|^{1/k_n} \to 1/R$.*

Then, the sequence $(\nu(f_n))$ of probability measures converges to the Dirac measure at the point $\xi(R)$.

Proof. — Let f be the limit of f_n; it is an analytic function on the disk $E(0,R)$, hence an element of the Tate algebra $K\{R^{-1}T\}$. Condition (1) of Theorem 2.1 is obviously verified.

Since $f \neq 0$ and $E(0,R)$ is compact and connected, the function f has only finitely many zeroes on $E(0,R)$, counted with multiplicities. Analogously to Hurwitz's theorem in complex analysis, Condition (2) of Theorem 2.1 also holds. Indeed, let us even show that $\nu(f_n)(E(0,R)) \to 0$. Up to replacing K by a complete algebraically closed extension, we may assume that $R = |a|^{-1}$ for some $a \in K^*$. Then, the theory of Newton polygons implies that $k_n\nu(f_n)(C)$ is the degree of the reduction of the polynomial $\widetilde{f_n(aT)}$. Clearly this degree converges to that of the polyonomial $\widetilde{f(aT)}$, so we obtain that $\nu(f_n)(E(0,R)) \to 0$.

Finally, Condition (3) also holds, with $S = \{\infty\}$. This implies that $\nu(f_n)$ converges to the Dirac measure at $\xi(R)$, as claimed. ☐

Remark 3.2. — In the complex setting, it would be sufficient to assume that the sequence (f_n) converges uniformly on compact subsets of the open disk of radius R, while in the p-adic case, we have to assume that the uniform convergence holds on the full closed disk. This discrepancy is due to the fact that the interior of the p-adic unit $E = E(0,R)$ disk is much larger than the open p-adic unit disk $D(0,R)$. In fact,

$\mathring{E}$ is the complement to the Gauss point $\xi(R)$ in E. This makes the equidistribution statement of Theorem 2.1 almost pointless in this particular case. Indeed, the easiest part of its proof shows that any limit measure is supported by E. And since its assumption (2) requires that any limit measure does not charge $\mathring{E}$, this forces the limit measure to be a Dirac mass at the Gauss point $\xi(R)$. This is however unavoidable, *cf.* Example 3.7.

The following corollaries are especially interesting under the supplementary assumption that the coefficients of the polynomials f_n belong to a locally compact subfield K_0 of K. They can be proved directly for elements of a Tate algebra $K\{R^{-1}T\}$ but we keep to our initial goal and view them as a consequence of the behavior of the limit measures of zeroes established in Proposition 3.1: they apply for any sequence (f_n) for which the sequence $(\nu(f_n))$ converges to the Dirac measure at a Gauß point $\xi(R)$.

Corollary 3.3. — *Let K_0 be a locally compact subfield of K. When $n \to \infty$, the number of K_0-roots of the polynomial f_n is $\mathrm{o}(k_n)$.*

Proof. — The conclusion is that $\mu_n(K_0) \to 0$. If it didn't hold, the limit measure of the sequence (μ_n) would charge K_0. But $\xi(R) \notin K_0$. □

Corollary 3.4. — *Let K_0 be a locally compact subfield of K. Let d be a positive integer. When $n \to \infty$, the number of irreducible factors in $K_0[T]$ of the polynomial f_n whose degree is $\leqslant d$ is $\mathrm{o}(k_n)$.*

Proof. — Replacing K by the completion of an algebraic closure, we assume that it is algebraically closed. Let $K_d \subset K$ be the extension of K_0 (in a fixed algebraic closure of K) generated by all roots of all polynomials of degree $\leqslant d$ in $K_0[T]$. It is well known that K_d is a finite extension of K_0. In particular, it is a locally compact subfield, hence the result follows from the first corollary. □

Corollary 3.5. — *Let K_0 be a locally compact subfield of K and assume that the polynomials f_n belong to $K_0[T]$. When $n \to \infty$, the maximal degree of an irreducible factor of f_n tends to $+\infty$.*

Proof. — Otherwise, up to replacing the sequence (f_n) by a subsequence of it, the irreducible factors of f_n would have a uniformly bounded degree, which contradicts the previous corollary. □

Let us explicit the particular case where, for each integer n, the polynomial f_n is the degree n truncation of a fixed power series $f = \sum_j a_j T^j$ with coefficients in a locally compact p-adic field.

Theorem 3.6. — *Let K be a finite extension of $\mathbf{Q}_p$ and let $f = \sum_{j=0}^{\infty} a_j T^j$ be a power series with coefficients in K. Let $R = (\overline{\lim}_j \left|a_j\right|^{1/j})^{-1}$ be its radius of convergence. Let us assume that $0 < R < \infty$ and that $f \in K\{R^{-1}T\}$.*

For each integer n, let $f_n = \sum_{j=0}^{n} a_j T^j$. Let $(n_k)_{k\geqslant 0}$ be a increasing sequence of integers such that $\left|a_{n_k}\right|^{1/n_k}$ tends to $1/R$ when $k \to \infty$. Then, the number of

irreducible factors of f_{k_n} of bounded degree is negligible before n_k, and the maximum degree of an irreducible factor of f_{n_k} tends to infinity when $k \to \infty$.

We conclude this paper by the following promised construction, which shows that in hypothesis (2) of Proposition 3.1, the closed disk $E(0, R)$ cannot be replaced by the open disk $D(0, R)$, and that Theorem 3.6 does not hold for arbitrary power series of radius of convergence R.

Example 3.7. — We begin with the simple following observation: for any degree m polynomial $f \in \mathbf{Z}[T]$, and any integer $n > m + 1$, there exists a unique monic polynomial $F \in \mathbf{Z}[T]$ of degree n such that $F \equiv f \mod T^{m+1}$ and $F \equiv 0 \mod (T-1)^{n-m-1}$. Indeed, write $F = f + T^{m+1}g(T-1) + T^n$, where the unknown polynomial $g \in \mathbf{Z}[T]$ has degree $\leqslant n - m - 2$; the condition on F translates into the condition that $g(T)$ is congruent modulo T^{n-m-1} to the power series with integer coefficients given by the expansion of $-(f(1+T) + (1+T)^n)/(1+T)^{m+1}$. The existence and uniqueness of F follows at once.

Thanks to this observation, we may construct by induction a sequence (F_n) of monic polynomials with integer coefficients such that $d_n = \deg(F_n) = 2^{n+1} - 2$ such that $F_{n+1}(T) \equiv F_n(T) \mod T^{d_n - 1}$ and $F_n(T)$ vanishes at order at least $2^n - 1$ at $T = 1$.

It follows that there exists a power series f such that, for any integer $n \geqslant 0$, the polynomial F_n is the truncation in degree d_n of f.

Fix a prime number p and view the power series f as a power series with p-adic coefficients. Its radius of convergence is equal to 1.

The sequence (F_n) satisfies Hypotheses (1) and (3) of Proposition 3.1. Moreover, F_n converges to f, uniformly on any compact subset of the open disk $D(0, 1)$. However, any limit measure ν of the sequence $\nu(F_n)$ satisfies $\nu \geqslant \frac{1}{2}\delta_1$. In particular, $\nu \neq \delta_{\xi(1)}$, so that (F_n) does not satisfies the conclusion of Proposition 3.1.

Moreover, for any n, F_n has at least $\frac{1}{2}\deg(F_n)$ irreducible factors of degree 1, so that the conclusion of Theorem 3.6 does not hold for f neither.

It remains an interesting open question to find more general hypotheses on a power series f of given radius of convergence R which would guarantee that, in adequate subsequences, the measures $\nu(f_n)$ equidistribute towards the Gauss point $\xi(R)$.

References

[1] Vladimir V. Andrievskii and Hans-Peter Blatt, *Discrepancy of signed measures and polynomial approximation*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2002.

[2] Matthew Baker and Robert Rumely, *Potential theory on the Berkovich projective line*, Mathematical Surveys and Monographs, vol. 159, Amer. Math. Soc., 2010.

[3] Vladimir G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990.

[4] Marcel Brelot, *Lectures on potential theory*, Tata Institute of Fundamental Research Lectures on Mathematics, no. 19, Tata Institute of Fundamental Research, Bombay, 1967, notes by K. N. Gowrisankaran and M. K. Venkatesha Murthy, second edition revised and enlarged with the help of S. Ramaswamy.

[5] Robert F. Coleman, *On the Galois groups of the exponential Taylor polynomials*, Enseign. Math. (2) **33** (1987), no. 3-4, 183–189.

[6] Charles Favre and Mattias Jonsson, *The valuative tree*, Lecture Notes in Mathematics, vol. 1853, Springer-Verlag, Berlin, 2004.

[7] Charles Favre and Juan Rivera-Letelier, *Equidistribution quantitative des points de petite hauteur sur la droite projective*, Math. Ann. **335** (2006), no. 2, 311–361, *arXiv:math.NT/0407471*.

[8] Robert Jentzsch, *Fortgesetzte Untersuchungen über die Abschnitte von Potenzreihen*, Acta Math. **41** (1916), no. 1, 253–270.

[9] Thomas Ransford, *Potential theory in the complex plane*, Cambridge University Press, Cambridge, 1995.

[10] Robert S. Rumely, *Capacity theory on algebraic curves*, Lecture Notes in Math., vol. 1378, Springer-Verlag, Berlin, 1989.

[11] Issai Schur, *Gleichungen ohne Affekt*, Sitzungsberichte Preuss. Akad. Wiss. Phys.-Math. Klasse (1930), 443–449, reproduced in *Gesammelte Abhandlungen*, p. 191–197.

[12] Gábor Szegő, *Über die Nullstellen von Polynomen, die in einem Kreis gleichmässig konvergieren*, Sitzungsber. Ber. Math. Ges. **21** (1922), 59–64.

[13] Amaury Thuillier, *Théorie du potentiel sur les courbes en géométrie non archimédienne. Applications à la théorie d'Arakelov*, Ph.D. thesis, Université de Rennes 1, 2005, *http://tel.archives-ouvertes.fr/tel-00010990/*.

[14] Matasugu Tsuji, *Potential theory in modern function theory*, Chelsea Publishing Co., New York, 1975, Reprinting of the 1959 original.